

# Universal graphs at the successor of a singular cardinal

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## Abstract

The paper is concerned with the existence of a universal graph at the successor of a strong limit singular  $\mu$  of cofinality  $\aleph_0$ . Starting from the assumption of the existence of a supercompact cardinal, a model is built in which for some such  $\mu$  there are  $\mu^{++}$  graphs on  $\mu^+$  that taken jointly are universal for the graphs on  $\mu^+$ , while  $2^{\mu^+} \gg \mu^{++}$ .

The paper also addresses the general problem of obtaining a framework for consistency results at the successor of a singular strong limit starting from the assumption that a supercompact cardinal  $\kappa$  exists. The result on the existence of universal graphs is obtained as a specific application of a more general method.<sup>1</sup>

## 0 Introduction

The question of the existence of a universal graph of certain cardinality and with certain properties has been the subject of much research in mathematics ([FuKo], [Kj], [KoSh 492], [Rd], [Sh 175a], [Sh 500]). By universality we mean here that every other graph of the same size embeds into the universal graph. In the presence of *GCH* it follows from the classical results in model theory ([ChKe]) that such a graph exists at every uncountable cardinality, and it is well known that the random graph ([Rd]) is universal for countable graphs (although the situation is not so simple when certain requirements on the graphs are imposed, see [KoSh 492]). When the assumption of *GCH* is dropped, it becomes much harder to construct universal objects, and it is in fact usually rather easy to obtain negative consistency results by adding Cohen subsets to the universe (see [KjSh 409] for a discussion of this). For some classes of graphs there are no universal objects as soon as *GCH* fails sufficiently ([Kj]), while for others there can exist consistently a small family of the class that acts jointly as a universal object for the class at the given cardinality ([Sh 457], [DjSh 614]). Much of what is known in the absence of *GCH* is known about successors of regular cardinals ([Sh 457], [DjSh 614]). In [Sh 175a] there is a positive consistency result concerning the existence of a universal graph at the successor of singular  $\mu$  where  $\mu$  is not a strong limit. In this paper we address the issue of the existence of a universal graph at the successor of a singular strong limit and obtain a positive consistency results

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regarding the existence of a small family of such graphs that act jointly as universal for the graphs of the same size.

In addressing this specific problem, the paper also offers a step towards the solution of a more general problem of doing iterated forcing in connection with the successor of a singular. This is the case because the result about universal graphs is obtained as an application of a more general method. The method relies on an iteration of  $(< \kappa)$ -directed-closed  $\theta \geq \kappa^+$ -cc forcing, followed by the Prikry forcing for a normal ultrafilter  $\mathcal{D}$  built by the iteration. The cardinal  $\kappa$  here is supercompact in the ground model. The idea is that the Prikry forcing for  $\mathcal{D}$  can be controlled by the iteration, as  $\mathcal{D}$  is being built in the process as the union of an increasing sequence of normal filters that appear during the iteration. Apart from building  $\mathcal{D}$ , the iteration also takes care of the particular application it is aimed at by predicting the  $\mathcal{D}$ -names of the relevant objects and taking care of them (in our application, these objects are graphs on  $\kappa^+$ ). The iteration is followed by the Prikry forcing for  $\mathcal{D}$ , so changing the cofinality of  $\kappa$  to  $\aleph_0$ . Before doing the iteration we prepare  $\kappa$  by rendering its supercompactness indestructible by  $(< \kappa)$ -directed-closed forcing through the use of Laver's diamond ([La]). Not only do we use the indestructibility of  $\kappa$ , but Laver's diamond itself plays a crucial role in the definition of the iteration. We note that the result has an unusual feature in which the iteration is not constructed directly, but the existence of such an iteration is proved and used.

Some of the ideas connected to the forcing scheme discussed in this paper were pursued by A. Mekler and S. Shelah in [MkSh 274], and by M. Gitik and S. Shelah in [GiSh 597], both in turn relaying on M. Magidor's independence proof for  $SCH$  at  $\beth_\omega$  [Ma 1], [Ma 2] and Laver's indestructibility method, [La]. In [MkSh 274]§3 the idea of guessing Prikry names of an object after the final collapse is present, while [GiSh 597] considers densities of box topologies, and for the particular forcing used there presents a scheme similar to the one we use (although the iteration is different). The latter paper also reduced the strength of a large cardinal needed for the iteration to a hyper-measurable. The difference between [GiSh 597] and our results is that the individual forcing used in [GiSh 597] is basically Cohen forcing, while our interest here is to give a general axiomatic framework under which

the scheme can be applied for many types of forcing notions.

The investigation of the consistent existence of universal objects also has relevance in model theory. The idea here is to classify theories in model theory by the size of their universality spectrum, and much research has been done to confirm that this classification is interesting from the model-theoretic point of view ([GrSh 174], [KjSh 409], [Sh 500], [DjSh 614]). The results here sound a word of caution to this programme. Our construction builds  $\mu^{++}$  graphs on  $\mu^+$  that are universal for the graphs on  $\mu^+$ , while  $2^{\mu^+} \gg \mu^{++}$  and  $\mu$  is a strong limit singular of cofinality  $\aleph_0$ . In this model we naturally obtain club guessing on  $S_{\aleph_0}^{\mu^+}$  for order type  $\mu$ , and this will prevent the prototype of a stable unsuperstable theory  $\text{Th}({}^\omega\omega, E_n)_{n<\omega}$  from having a small universal family, see [Sh 457], [KjSh 447]. Hence the universality spectrum at such  $\mu^+$  classifies the prototype of a simple unstable theory (the theory of a random graph), as less complicated than the prototype of a stable unsuperstable theory, contrary to the expectation. A possible conclusion is that one should concentrate the investigation of the universality spectrum as a dividing line for unstable theories only on the case  $\lambda^+$  with  $\lambda = \lambda^{<\lambda}$ , as the case of the successor of a singular is too sensitive to the set theory involved.

There are several further questions that this paper brings to mind. From the point of view of model theory it would be interesting to determine which other first order theories fit the scheme of this paper and from the point of view of graph theory one would like to improve the result on the existence of  $\mu^{++}$  jointly universal graphs to having just one universal graph. Set-theoretically, we would like to be able to replace  $\mu$  an unspecified singular strong limit by  $\mu = \beth_\omega$ , as well as to investigate singulars of different cofinality than  $\aleph_0$ . We did not concentrate here on obtaining the right consistency strength for our results, suggesting another question that may be addressed in the future work.

The paper is organised as follows. The major issue is to define the iteration used in the second step of the above scheme, which is done in certain generality in §1. We give there a sufficient condition for a one step forcing to fit the general scheme, so obtaining an axiomatic version of the method. In §2 we give the application to the existence of  $\mu^{++}$  universal graphs of size  $\mu^+$  for  $\mu$  the successor of a strong limit singular of cofinality  $\aleph_0$ .

Most of our notation is entirely standard, with the possible exception of

**Notation 0.1.** For  $\alpha$  and ordinal and a regular cardinal  $\kappa < \alpha$ , we let

$$S_\kappa^\alpha \stackrel{\text{def}}{=} \{\beta < \alpha : \text{cf}(\beta) = \kappa\}.$$

## 1 The general framework for forcing

**Definition 1.1.** Suppose that  $\kappa$  is a strongly inaccessible cardinal  $> \aleph_0$ . A function  $h : \kappa \rightarrow \mathcal{H}(\kappa)$  is called *Laver's diamond on  $\kappa$*  iff for every  $x$  and  $\lambda$ , there is an elementary embedding  $\mathbf{j} : V \rightarrow M$  with

- (1)  $\text{crit}(\mathbf{j}) = \kappa$  and  $\mathbf{j}(\kappa) > \lambda$ ,
- (2)  ${}^\lambda M \subseteq M$ ,
- (3)  $(\mathbf{j}(h))(\kappa) = x$ .

**Theorem 1.2.** Laver ([La]) Suppose that  $\kappa$  is a supercompact cardinal. Then there is a Laver's diamond on  $\kappa$ .

**Hypothesis 1.3.** We work in a universe  $V$  that satisfies

- (1)  $\kappa$  is a supercompact cardinal,  $\theta = \text{cf}(\theta) \geq \kappa^+$  and *GCH* holds at and above  $\kappa$ ,
- (2)  $\Upsilon^\theta = \Upsilon$  &  $\chi = \Upsilon^+$  and
- (3)  $h : \kappa \rightarrow \mathcal{H}(\kappa)$  is a Laver's diamond.

**Remark 1.4.** It is well known that the consistency of the above hypothesis follows from the consistency of the existence of a supercompact cardinal. We in fact only use the  $\chi$ -supercompactness of  $\kappa$ .

**Definition 1.5.** Laver ([La]) We define

$$\bar{R} = \langle R_\alpha^+, R_\beta : \alpha \leq \kappa, \beta < \kappa \rangle,$$

an iteration done with Easton supports, and a strictly increasing sequence  $\langle \lambda_\alpha : \alpha < \kappa \rangle$  of cardinals, where  $R_\alpha$  and  $\lambda_\alpha$  are defined by induction on  $\alpha < \kappa$  as follows.

If

- (1)  $h(\alpha) = (P, \lambda)$ , where  $\lambda$  is a cardinal and  $P$  is a  $R_\alpha^+$ -name of  $(< \alpha)$ -directed-closed forcing, and
- (2)  $(\forall \beta < \alpha) [\lambda_\beta < \alpha]$ ,

we let  $R_\alpha \stackrel{\text{def}}{=} P$  and  $\lambda_\alpha \stackrel{\text{def}}{=} \lambda$ . Otherwise, let  $R_\alpha \stackrel{\text{def}}{=} \{\emptyset\}$  and  $\lambda_\alpha \stackrel{\text{def}}{=} \sup_{\beta < \alpha} \lambda_\beta$ .

The extension in  $R_\alpha^+$  is defined by letting

$$p \leq q \iff [\text{Dom}(p) \subseteq \text{Dom}(q) \ \& \ (\forall i \in \text{Dom}(q))(q \restriction i \Vdash "p(i) \leq q(i)"),$$

(where  $p$  denotes the weaker condition).

**Remark 1.6.** The forcing  $R_\kappa^+$  used in this section is Laver's forcing from [La] which makes the supercompactness of  $\kappa$  indestructible under any  $(< \kappa)$ -directed-closed forcing.

**Convention 1.7.** Definitions 1.8 and 1.11, Claim 1.12 and Observation 1.13 take place in  $V_1 \stackrel{\text{def}}{=} V^{R_\kappa^+}$ . Notice that  $\kappa^+ \leq \text{cf}(\theta) = \theta < \chi$  still holds in  $V_1$ , as  $\text{Rang}(h) \subseteq \mathcal{H}(\kappa)$ , and that  $\kappa$  is still supercompact.

**Definition 1.8.** We define the family  $\mathcal{K}_\theta$  as the family of all sequences

$$\bar{Q} = \langle P_i, Q_i, A_i : i < i^* = \text{lg}(\bar{Q}) < \chi \rangle$$

which satisfy

- (1)  $P_i \subseteq \mathcal{H}(\chi)$  (and each  $P_i$  is a forcing notion, which will follow from the rest of the definition),

- (2)  $\langle P_i : i < i^* \rangle$  is  $\triangleleft$ -increasing and each satisfies  $\chi$ -cc,
- (3)  $Q_i$  is a  $P_i$ -name of a member of  $\mathcal{H}(\chi)$  (hence of cardinality  $\leq \Upsilon$ ),
- (4) If  $\text{cf}(i) \geq \kappa$ , then  $P_i = \bigcup_{j < i} P_j$ ,
- (5)  $A_i$  is a canonical  $P_{i+1}$ -name of a subset of  $\kappa$ ,
- (6) Letting  $G_i$  be  $P_i$ -generic over  $V_1$ , then in  $V_1[G_i]$ ,
- (a)  $\text{NUF} \stackrel{\text{def}}{=} \{\mathcal{D} : \mathcal{D} \text{ a normal ultrafilter on } \kappa\}$ ,
- (b) for every  $\mathcal{D} \in \text{NUF}$  we are given a  $(< \kappa)$ -directed-closed forcing notion  $Q_{\mathcal{D}}^i \in \mathcal{H}(\chi)^{V_1[G_i]}$  whose minimal element is denoted by  $\emptyset_{Q_{\mathcal{D}}^i}$ ,
- (7) With the notation of (6), we have that  $Q_i[G_i]$  is

$$\{\emptyset\} \cup \text{NUF} \cup \{\{\mathcal{D}\} \times Q_{\mathcal{D}}^i : \mathcal{D} \in \text{NUF}\}.$$

- (8) The order on  $Q_i[G_i]$  is given by letting

$$x \leq y \text{ iff } [x = y \text{ or } x = \emptyset \text{ or } (x = \mathcal{D} \in \text{NUF} \ \& \ y \in \{x\} \times Q_{\mathcal{D}}^i) \text{ or}$$

$$x = (\mathcal{D}, x^*), y = (\mathcal{D}, y^*) \text{ for some } \mathcal{D} \in \text{NUF} \text{ and } Q_{\mathcal{D}}^i \models "x^* \leq y^*"],$$

- (9) We have

$$P_i \stackrel{\text{def}}{=} \left\{ p : \begin{array}{l} \text{(i) } p \text{ is a function with domain } \subseteq i, \\ \text{(ii) } j \in \text{Dom}(p) \implies p(j) \text{ is a canonical } P_j\text{-name} \\ \text{of a member of } Q_j \\ \text{(iii) } |S\text{Dom}(p)| < \kappa \text{ (see below)} \end{array} \right\},$$

ordered by letting

$$p \leq q \iff [\text{Dom}(p) \subseteq \text{Dom}(q) \ \& \ (\forall i \in \text{Dom}(q))(q \restriction i \Vdash p(i) \leq q(i))],$$

where

(Definition 1.8 continues below)

**Notation 1.9.** (A) For  $i < i^*$ , and  $p \in P_i$ , we let

$$S\text{Dom}(p) \stackrel{\text{def}}{=} \left\{ j \in \text{Dom}(p) : \neg \left[ p \restriction j \Vdash_{P_j} \begin{array}{l} \text{“}p(j) \in \{\emptyset\} \cup \text{NUF}\text{”} \\ \cup \{(\mathcal{D}, \emptyset_{\mathcal{D}}^j) : \mathcal{D} \in \text{NUF}\} \end{array} \right] \right\},$$

(B) For  $i < i^*$  and  $p \in P_i$  we call  $p$  *purely full* iff:

$S\text{Dom}(p) = \emptyset$  and for every  $j < i$  we have

$$p \restriction j \Vdash_{P_j} \text{“}p(j) \in \text{NUF}\text{”}.$$

(C) Suppose that  $i < i^*$  and  $p \in P_i$  is purely full, we define

$$P_i/p \stackrel{\text{def}}{=} \{q \in P_i : q \geq p \text{ \& each } q(j) \text{ has form } (\mathcal{D}, x) \text{ for some } x\},$$

with the order inherited from  $P_i$ .

(*Definition 1.8 continues:*)

(10) For every  $i \leq i^*$  and  $p \in P_i$  which is purely full we have that  $P_i/p$  satisfies  $\theta$ -cc and  $P_i/p \in \mathcal{H}(\chi)$ .

**Observation 1.10.** (1) If  $\bar{Q} \in \mathcal{K}_\theta$  and  $i < \text{lg}(\bar{\theta})$ , then  $P_{i+1} = P_i * Q_i$ .

(2) Assuming that  $\langle P_j, \mathcal{Q}_j, \mathcal{A}_j : j < i \rangle \in \mathcal{K}_\theta$  and (5)-(10) above hold, we can see that  $\langle P_j, \mathcal{Q}_j, \mathcal{A}_j : j \leq i \rangle \in \mathcal{K}_\theta$ . Hence  $\mathcal{K}_\theta$  can be alternatively defined by specifying when  $\langle P_j, \mathcal{Q}_j, \mathcal{A}_j : j < i \rangle \in \mathcal{K}_\theta$  by induction on  $i < \chi$ .

**Definition 1.11.** (1) Let  $\kappa^+ \leq \text{cf}(\theta) = \theta < \chi$ . We define the family  $\mathcal{K}_\theta^+$  as the family of all sequences

$$\bar{Q} = \langle P_i, \mathcal{Q}_i, \mathcal{A}_i : i < \chi \rangle$$

such that

$$i < \chi \implies \bar{Q} \restriction i \in \mathcal{K}_\theta.$$

We let  $P_\chi \stackrel{\text{def}}{=} \bigcup_{i < \chi} P_i$ .



- (2) Suppose that  $\bar{Q} \in \mathcal{K}_\theta^+$  and  $\langle p_i : i < \chi \rangle$  with  $p_i \in P_{\zeta_i}$  are purely full and increasing in  $P_\chi$ , where  $\zeta_i \stackrel{\text{def}}{=} \min\{\zeta : p_i \in P_\zeta\}$  (so if  $i < j$  then  $p_i = p_j \restriction \zeta_i$ ). We define

$$P_\chi / \cup_{i < \chi} p_i \stackrel{\text{def}}{=} \{q \in P_\chi : (\forall i < \chi)[q \restriction \zeta_i \in P_{\zeta_i}/p_i]\},$$

with the order inherited from  $P_\chi$ .

**Claim 1.12.** (1) If  $\bar{Q} \in \mathcal{K}_\theta$ , then for all  $i \leq \text{lg}(\bar{Q})$ , we have that  $P_i$  is  $(< \kappa)$ -directed-closed.

- (2) Similarly for  $\bar{Q} \in \mathcal{K}_\theta^+$ .

**Proof of the Claim.** (1) Given a directed family  $\{p_\alpha : \alpha < \alpha^* < \kappa\}$  of conditions in  $P_i$ . We shall define a common extension  $p$  of this family. Let us first let  $\text{Dom}(p) \stackrel{\text{def}}{=} \bigcup_{\alpha < \alpha^*} \text{Dom}(p_\alpha)$ . For  $j \in \text{Dom}(p)$ , we define  $p(j)$  by induction on  $j$ . We work in  $V_1^{P_j}$  and assume that  $\{p_\alpha \restriction j : \alpha < \alpha^*\} \subseteq G_{P_j}$ .

If  $j \notin \bigcup_{\alpha < \alpha^*} S\text{Dom}(p_\alpha)$ , then notice that there is at most one  $\mathcal{D} \neq \emptyset$  such that for some (possibly more than one)  $\alpha < \alpha^*$  we have  $p_\alpha \restriction j \Vdash "p_\alpha(j) = \mathcal{D}"$ , as the family is directed. If there is such  $\mathcal{D}$ , we let  $p(j) \stackrel{\text{def}}{=} \mathcal{D}$ , otherwise we let  $p(j) = \emptyset$ .

If  $j \in \bigcup_{\alpha < \alpha^*} S\text{Dom}(p_\alpha)$ , similarly to the last paragraph, we conclude that there is exactly one  $\mathcal{D}$  such that

$$[\alpha < \alpha^* \ \& \ j \in S\text{Dom}(p_\alpha)] \implies p_\alpha \restriction j \Vdash "p_\alpha(j) \in \{\mathcal{D}\} \times Q_{\mathcal{D}}^j".$$

As  $Q_{\mathcal{D}}^j$  is forced to be  $(< \kappa)$ -directed-closed, we can find in  $V_1^{P_j}$  a condition  $q$  such that  $q \geq p_j(\alpha)$  for all  $\alpha < \alpha^*$  such that  $j \in S\text{Dom}(p_\alpha)$ . Let  $p(j) \stackrel{\text{def}}{=} q$  for some such  $q$ .

- (2) Follows from (1) as  $\chi = \text{cf}(\chi) > \kappa$ . ★<sub>1.12</sub>

**Observation 1.13.** Suppose that  $\bar{Q} \in \mathcal{K}_\theta^+$ ,  $i < j < \chi$  and  $p \in P_i$ ,  $q \in P_j$  are purely full, while  $p \leq q$ . Then

- (1)  $\text{Dom}(p) \subseteq \text{Dom}(q)$  and  $\alpha \in \text{Dom}(p) \implies p(\alpha) = q(\alpha)$ .

(2) Suppose that  $r \in P_i/p$ .

Then defining  $r + q \in P_j$  by letting  $\text{Dom}(r + q) = \text{Dom}(q)$  and letting for  $\alpha \in \text{Dom}(r)$

$$(r + q)(\alpha) \stackrel{\text{def}}{=} \begin{cases} r(\alpha) & \text{if } \alpha \in \text{Dom}(p) \\ (q(\alpha), \emptyset_{Q_{q(\alpha)}^\alpha}) & \text{otherwise,} \end{cases}$$

we obtain a condition in  $P_j/q$ .

(3) For  $r_1, r_2 \in P_i/p$  we have that

- ( $\alpha$ )  $r_1$  and  $r_2$  are incompatible in  $P_i/p$  iff  $r_1 + q$  and  $r_2 + q$  are incompatible in  $P_j/q$ ,
- ( $\beta$ )  $r_1 \leq_{P_i/p} r_2 \iff r_1 + q \leq_{P_j/q} r_2 + q$ .

(4)  $P_i/p \triangleleft_f P_j/q$  where  $f(r) \stackrel{\text{def}}{=} r + q$ .

(5) Suppose that the sequence  $\bar{p} = \langle p_i : i < \chi \rangle$  satisfies that each  $p_i \in P_{\zeta_i}$  is purely full, and the sequence  $\bar{p}$  is increasing in  $P_\chi$ , where

$$\zeta_i \stackrel{\text{def}}{=} \min\{\zeta : p_i \in P_\zeta\} = \xi_i + 1 > i$$

and  $\langle \zeta_i : i < \chi \rangle$  is strictly increasing. Then  $P^* = P_\chi / \cup_{i < \chi} p_i$  is isomorphic to the limit of a  $(< \kappa)$ -supported iteration of  $(< \kappa)$ -directed-closed  $\theta$ -cc forcing, namely

$$P^* \approx \lim \langle P_{\xi_i} / (p_i \restriction \xi_i), Q_{p_i(\xi_i)}^{\xi_i} : i < \chi \rangle,$$

with the complete embeddings  $f_{p_i, p_j} : P_{\xi_i} / (p_i \restriction \xi_i) \rightarrow P_{\xi_j} / (p_j \restriction \xi_j)$  as in (4) above.

(6) For every  $r \in P_\chi$ , there is  $q \geq r$  with  $S\text{Dom}(q) = S\text{Dom}(r)$  and  $p$  purely full in some  $P_i$ , such that  $q \in P_i/p$ .

**Convention 1.14.** Since  $f_{p_i, p_j}$  are usually clear from the context we simplify the notation by not mentioning these functions explicitly.

**Claim 1.15.** Suppose that  $\bar{Q} \in \mathcal{K}_\theta^+$  and  $\underline{t}$  is a  $P_\chi$ -name of an ordinal, while  $p \in P_\chi$  is purely full.

Then for some  $j < \chi$  and  $q$  we have  $p \leq q \in P_j$ , and  $q$  is purely full, and above  $q$  we have that  $\underline{t}$  is a  $P_j$ -name (i.e  $\underline{t}$  is a  $P_j/q$ -name).

**Proof of the Claim.** Given  $p \in P_\chi$  purely full, and suppose that the conclusion fails. Let  $i < \chi$  be such that  $p \in P_i$ . We shall choose by induction on  $\zeta < \theta$  ordinals  $i_\zeta$  and  $\gamma_\zeta$  and condition  $r_\zeta$  such that

- (i)  $i_\zeta \in [i, \chi)$  and  $\langle \zeta : \zeta < \theta \rangle$  is increasing continuous,
- (ii)  $p_\zeta \in P_{i_\zeta}$  is purely full, with  $p_0 = p$ ,
- (iii)  $p_\zeta \leq r_\zeta$  with  $r_\zeta \Vdash_{P_\chi} \text{"}\underline{t} = \gamma_\zeta\text{"}$ ,
- (iv)  $\gamma_\zeta \notin \{\gamma_\xi : \xi < \zeta\}$ , so in particular  $r_\zeta$  is incompatible with every  $r_\xi$  for  $\xi < \zeta$ ,
- (v)  $p_\zeta \stackrel{\text{def}}{=} \cup_{\xi < \zeta} p_\xi$  for  $\zeta$  a limit.
- (vi)  $r_\zeta \in P_{i_{\zeta+1}}/p_{\zeta+1}$ .

We now explain how to do this induction.

Given  $p_\zeta$  and  $i_\zeta$ . Since we are assuming that  $\underline{t}$  is not a  $P_{i_\zeta}$ -name above  $p_\zeta$ , it must be possible to find  $r_\zeta$  and  $\gamma_\zeta$  as required. Having chosen  $r_\zeta$ , (by extending  $r_\zeta$  if necessary), we can choose  $p_{\zeta+1}$  as required in item (vi) above, see Observation 1.13(6). This determines  $i_{\zeta+1}$ . Note that  $i_{\zeta+1} < \chi$  as  $P_\chi \stackrel{\text{def}}{=} \bigcup_{j < \chi} P_j$ .

However, completing the induction we arrive at a contradiction, as letting  $p^* \stackrel{\text{def}}{=} \cup_{\zeta < \theta} p_\zeta$  we obtain a purely full condition. Hence  $P \stackrel{\text{def}}{=} P_{\sup_{\zeta < \theta} i_\zeta}/p^*$  has  $\theta$ -cc, but  $\{r_\zeta + p^* : \zeta < \theta\}$  forms a set of  $\theta$  pairwise incompatible conditions in  $P$ .  $\star_{1.15}$

**Convention 1.16.** Now we go back to  $V$ , i.e. the Main Claim 1.17 takes place in  $V$ .

**Main Claim 1.17.** Suppose

- ( $\alpha$ )  $\bar{Q} = \langle \bar{P}_i, \bar{Q}_i, \bar{A}_i : i < \chi \rangle$  is an  $R_\kappa^+$ -name for a member of  $\mathcal{K}_\theta^+$ ,
- ( $\beta$ )  $\mathbf{j} : V \rightarrow M$  is an elementary embedding such that  ${}^\Upsilon M \subseteq M$ ,  $\text{crit}(\mathbf{j}) = \kappa$ ,  
 $\chi < \mathbf{j}(\kappa)$  and

$$(\mathbf{j}(h))(\kappa) = (\bar{P}_\chi, \chi)$$

(such a choice is possible by the definition of Laver's diamond) .

Considering  $\mathbf{j}(\langle R_\alpha^+, \bar{R}_\alpha : \alpha < \kappa \rangle)$  in  $M$ , by its definition we see that

$$\mathbf{j}(\langle R_\alpha^+, \bar{R}_\alpha : \alpha < \kappa \rangle) = \langle R_\alpha^+, \bar{R}_\alpha : \alpha < \mathbf{j}(\kappa) \rangle$$

and  $\bar{R}_\kappa = \bar{P}_\chi$ . Hence  $\mathbf{j}(R_\kappa^+) = R_\kappa^+ * \bar{P}_\chi * \bar{R}^*$  for some  $R_\kappa^+ * \bar{P}_\chi$ -name  $\bar{R}^* \in M$  for a forcing notion, which is forced to be  $\chi^+$ -closed.

We also let

$$\bar{Q}' = \langle \bar{P}'_i, \bar{Q}'_i, \bar{A}'_i : i < \mathbf{j}(\chi) \rangle \stackrel{\text{def}}{=} \mathbf{j}(\langle \bar{P}_i, \bar{Q}_i, \bar{A}_i : i < \chi \rangle).$$

Then in  $V^{R_\kappa^+}$ , the following holds: we can find  $\bar{\alpha} = \langle \alpha_i : i < \chi \rangle$ ,  $\bar{p}^* = \langle p_i^* : i < \chi \rangle$  and  $\bar{q}^* = \langle q_i^* = \langle {}^1q_i, {}^2q_i \rangle : i < \chi \rangle$  such that

- (a)  $\langle \alpha_i : i < \chi \rangle$  is strictly increasing continuous and each  $\alpha_i < \chi$ ,
- (b)  $p_i^* \in P_{\alpha_i+1}$  is purely full,
- (c)  $\bar{p}^*$  is increasing in  $P_\chi$ ,
- (d) For every  $i < \chi$ , we have  $\bar{q}^* \restriction i \in M^{R_\kappa^+}$ , and in  $M^{R_\kappa^+}$  we have

$$(p_i^*, {}^1q_i, {}^2q_i) \in P_\chi * \bar{R}^* * \bar{P}'_{\mathbf{j}(\alpha_i+1)},$$

while  $(p_i^*, {}^1q_i) \in P_\chi * \bar{R}^*$ ,

- (e) In  $M^{R_\kappa^+}$  we have that for  $\gamma < \chi$

$$\langle (p_i^*, {}^1q_i, {}^2q_i) : i < \gamma \rangle \text{ is increasing in } P_\chi * \bar{R}^* * \bar{P}'_{\sup_{i < \gamma} \mathbf{j}(\alpha_i+1)},$$

(f) In  $M^{R_\kappa^+}$ , it is forced by  $(p_{i+1}^*, {}^1q_{i+1})$  that  ${}^2q_{i+1}$  is an upper bound to

$$\{\mathbf{j}(r) : r \in G_{P_{\alpha_i} * \dot{\mathcal{Q}}_{p_i^*}^{\alpha_i+1}}\},$$

(g) If  $\dot{B}$  is an  $R_\kappa^+$ -name of a  $P_{\alpha_i+1}$ -name of a subset of  $\kappa$ , then for some  $R_\kappa^+ * P_\chi$ -name  $\dot{\mathbf{t}}_B$  for a truth value (i.e. an ordinal  $\in \{0, 1\}$ ):

- (1) In  $V$  we have that  $(\emptyset_{R_\kappa^+}, p_{i+1}^*)$  forces  $\dot{\mathbf{t}}_B$  to be a  $P_{\alpha_i+1+1}/p_{i+1}^*$ -name,
- (2)  $M \models [(\emptyset_{R_\kappa^+}, p_{i+1}^*, q_{i+1}^*) \Vdash \text{“}\kappa \in \mathbf{j}(\dot{B}) \text{ iff } \dot{\mathbf{t}}_B = 1\text{”}]$ .

(i) In  $M^{R_\kappa^+}$ , either

$$(p_{i+1}^*, q_{i+1}^*) \Vdash \text{“}\kappa \in \mathbf{j}(\dot{A}_{\alpha_i})\text{”},$$

or:

$$\begin{aligned} & \text{“ there is no } q = ({}^1q, {}^2q) \geq_{R^* * \dot{\mathcal{P}}_{\mathbf{j}(\alpha_i)+1}'} q_i^* \text{ with} \\ & p_i^* \Vdash_{P_\chi} {}^1q \Vdash_{R^*} \text{“} {}^2q(\mathbf{j}(\alpha_i)) \geq_{\dot{\mathcal{P}}_{\mathbf{j}(\alpha_i)+1}'} \{\mathbf{j}(r) : r \in G_{P_{\alpha_i} * \dot{\mathcal{Q}}_{p_i^*}^{\alpha_i}}\} \\ & \text{and } \kappa \in \mathbf{j}(\dot{A}_{\alpha_i})\text{”} \end{aligned}$$

[Note that  $\mathbf{j}(\dot{A}_{\alpha_i})$  is a  $\dot{\mathcal{P}}_{\mathbf{j}(\alpha_i)+1}'$ -name for a subset of  $\mathbf{j}(\kappa)$ .]

(j) If  $\text{cf}(i) \geq \theta$ , then in  $V^{R_\kappa^+ * \dot{\mathcal{P}}_{\alpha_i}}$  we have  $p_i^*(\alpha_i) \in NUF$  and specifically

$$p_i^*(\alpha_i) = \left\{ \dot{B}[G_{P_{\alpha_i}}] : \begin{array}{l} \dot{B} \text{ is a } P_{\alpha_i}/(p_i^* \restriction \alpha_i)\text{-name for a subset of } \kappa \\ \text{and } \dot{\mathbf{t}}_B[G_{P_{\alpha_i}}] = 1 \end{array} \right\}.$$

**Remark 1.18.** In fact, to accommodate various applications, we might want to weaken item (j) of the Main Claim 1.17, say to apply only to stationary many  $i \in S_{\geq \theta}^\chi$ . The same proof would work, but as we do not need this at present, we shall not go into this generality.

**Proof of the Main Claim.** Consider  $\langle R_i^+, \dot{R}_i : \kappa < i < \mathbf{j}(\kappa) \rangle$  over  $R_\kappa^+ * \dot{\mathcal{P}}_\chi$  in  $M$ . By the inductive definition of  $R_i$  (which is preserved by  $\mathbf{j}$ ), for  $\kappa < i < \chi^+$ , we have that  $\dot{R}_i$  is a name for the trivial forcing. For  $\chi^+ < i < \mathbf{j}(\kappa)$ , we

have that  $\tilde{R}_i$  is a name for a  $(< \chi)$ -directed-closed forcing in  $M$ , so in  $V$  as well, as  ${}^{<\chi}M \subseteq M$ . Similarly we conclude that  $\tilde{P}'_{\mathbf{j}(\zeta)}$  names a  $(< \chi)$ -closed forcing notion, for all  $\zeta < \chi$ . This observation will be used repeatedly and in particular will enable us to use the master condition idea in the induction below. In particular, we can conclude that  $\tilde{R}_\chi$  is  $(< \chi)$ -complete. By the choice of  $\mathbf{j}$ ,

$$\Vdash_{\mathbf{j}(R_\kappa^+)} \text{“each } P'_i/p \text{ is } (< \chi)\text{-closed for } p \in P'_i \text{ purely full.”}$$

Also notice that in the induction below, we have that in  $V_1$ , the cardinality of  $P_{\alpha_i+1}/p_i^*$  is  $\leq \Upsilon$  and  $P_{\alpha_i+1}/p_i^*$  satisfies  $\theta$ -cc, so in  $V_1^{P_{\alpha_i}}$  we have  $2^\kappa \leq \Upsilon$ .

Now we choose  $(\alpha_i, p_i^*, q_i^*)$  in  $M^{R_\kappa^+}$  by induction on  $i$ . We start with  $\alpha_0 = 0$ ,  $p_0^* \in P_1$  any purely full condition, and  $q_0^* = \emptyset$ .

Choice of  $p_{i+1}^*, q_{i+1}^*$  and  $\alpha_{i+1}$ .

Given  $p_i^*$  and  $\alpha_i$  in  $V^{R_\kappa^+}$ . We have that

$$p_i^* \restriction \alpha_i \Vdash_{P_{\alpha_i}} “|Q_{\tilde{p}_i^*(\alpha_i)}^{\alpha_i}| \leq \Upsilon \ \& \ G_{Q_{\tilde{p}_i^*(\alpha_i)}^{\alpha_i}} \subseteq P_{\alpha_i+1}/p_i^*.”$$

Hence in  $M$ , letting  $\tilde{X}_i \stackrel{\text{def}}{=} \{\mathbf{j}(r) : r \in G_{P_{\alpha_i} * Q_{\tilde{p}_i^*(\alpha_i)}^{\alpha_i}}\}$  we have

$$\begin{aligned} & \text{“}\tilde{X}_i \subseteq P'_{\mathbf{j}(\alpha_i)+1} \text{ is directed} \\ & (\emptyset_{R_{\mathbf{j}(\kappa)}^+}, \mathbf{j}(p_i^* \restriction \alpha_i)) \Vdash_{P'_{\mathbf{j}(\alpha_i)}} \text{and above} \\ & \mathbf{j}(p_i(\alpha_i)) \text{ has size } \leq \Upsilon.” \end{aligned}$$

In  $V_1$ , we have that the forcing  $P_{\alpha_i+1}/p_i^*$  is a  $\theta$ -cc forcing notion of size  $\leq \Upsilon$ , hence there are  $\leq \Upsilon^\theta \cdot \Upsilon = \Upsilon$  canonical  $P_{\alpha_i+1}/p_i^*$ -names for a subset of  $\kappa$ . Let us enumerate them as  $\langle \tilde{B}_\zeta^{i+1} : \zeta < \zeta^*(i+1) \leq \Upsilon \rangle$ , with  $\tilde{B}_0^{i+1} = \tilde{A}_i$ . By induction on  $\zeta \leq \zeta^*(i+1)$  we choose purely full  $p_\zeta^{i+1}$  increasing continuous with  $\zeta$ ,  $q_\zeta^{i+1} = ({}^1q_\zeta^{i+1}, {}^2q_\zeta^{i+1})$  increasing with  $\zeta$ ,  $\alpha_\zeta^{i+1}$  increasing with  $\zeta$  and  $\mathbf{t}_{\tilde{B}_\zeta^{i+1}}^{i+1}$  as follows.

Let  $p_0^{i+1} \stackrel{\text{def}}{=} p_i^*$ ,  $\alpha_0^{i+1} \stackrel{\text{def}}{=} \alpha_i$  and  $q_0^{i+1} \stackrel{\text{def}}{=} q_i^*$ .

Coming to  $\zeta + 1$ , let  $G$  be a  $R_\kappa^+ * P_\chi$  generic such that  $(\emptyset_{R_\kappa^+}, p_\zeta^{i+1}) \in G$  and let  $H$  be a  $\mathbf{j}(R_\kappa^+ * P_\chi)$  generic over  $M$  so that  $\{\mathbf{j}(r) : r \in G\} \subseteq H$ . This can be achieved by the familiar argument using the fact that  $\mathbf{j}(R_\kappa^+ * P_\chi)$  is  $(< \mathbf{j}(\kappa))$ -directed-closed, while  $G$  is  $(< \kappa)$ -directed and has size  $\leq \chi < \mathbf{j}(\kappa)$ .

In particular,  $\mathbf{j}$  lifts to an embedding of  $V[G] \rightarrow M[H]$ . By the fact that  $\mathbf{j}(R_\kappa^+ * \dot{P}_\chi) = R_\kappa^+ * \dot{P}_\chi * \dot{R}^* \dot{P}'_{\mathbf{j}(\chi)}$ , we can write  $M[H] = M[H_0][H_1]$  where  $H_0$  is  $R_\kappa^+ * \dot{P}_\chi$  generic over  $M$ . In  $M[H_0]$  we ask “the  $\zeta$ -question”:

Is it true that there is no

$$q = ({}^1q, {}^2q) \geq_{R^* * \dot{P}'_{\mathbf{j}(\alpha_\zeta^{i+1})+1}} \{q_\xi^{i+1} : \xi \leq \zeta\}$$

with

$${}^1q \Vdash_{R^*} “{}^2q \geq \dot{X}_i \ \& \ \kappa \in \mathbf{j}(B_\zeta^{i+1})” \ \& \ {}^2q \in \dot{P}'_{\mathbf{j}(\alpha_\zeta^{i+1})+1} / (\mathbf{j}(p_\zeta^{i+1}) \restriction \mathbf{j}(\alpha_\zeta^{i+1}) + 1)?$$

If the answer is positive, in  $M$  we define  $\dot{\mathbf{t}}_{B_\zeta^{i+1}} \stackrel{\text{def}}{=} 0$  (hence a  $R_\kappa^+ * \dot{P}_\chi$ -name for a truth value), and

$$q_{\zeta+1}^{i+1} = ({}^1q_{\zeta+1}^{i+1}, {}^2q_{\zeta+1}^{i+1})$$

to be any  $R_\kappa^+ * \dot{P}_\chi$ -name for a condition in  $\dot{R}^* * \dot{P}'_{\mathbf{j}(\chi)}$  such that

$$(\emptyset_{R_\kappa^+}, p_\zeta^{i+1}) \Vdash “{}^1q_{\zeta+1}^{i+1} \geq_{R^*} {}^1q_\xi^{i+1}”$$

for every  $\xi \leq \zeta$ , and

$$(\emptyset_{R_\kappa^+}, p_\zeta^{i+1}, {}^1q_{\zeta+1}^{i+1}) \Vdash_{\dot{P}'_{\mathbf{j}(\alpha_\zeta^{i+1})+1}} “{}^2q_{\zeta+1}^{i+1} \geq \cup \{{}^2q_\xi^{i+1} : \xi \leq \zeta\} \ \& \ {}^2q_{\zeta+1}^{i+1} \geq \dot{X}_i”.$$

The choice of  ${}^1q_{\zeta+1}^{i+1}$  is possible by the induction hypothesis and the fact that

$$\Vdash_{R_\kappa^+ * \dot{P}_\chi} “\dot{R}^* \text{ is } (< \chi)\text{-directed-closed}”.$$

Let us verify that the choice of  ${}^2q_{\zeta+1}^{i+1}$  is possible. Working in  $M$  we have that  $(\emptyset_{R_\kappa^+}, p_\zeta^{i+1}, {}^1q_{\zeta+1}^{i+1})$  forces  $\dot{X}_i$  over  $(\emptyset_{R_\kappa^+}, p_\zeta^{i+1}, {}^1q_{\zeta+1}^{i+1})$  to be a  $(< \kappa)$ -directed subset of  $\dot{P}'_{\mathbf{j}(\chi)}$  of size  $\leq \chi$ . Hence if  $\zeta = 0$  we can choose  ${}^2q^{i+1}$  to be forced to be above  $\dot{X}_i$ . We can similarly choose  ${}^2q_{\zeta+1}^{i+1}$  for  $\zeta > 0$ .

If the answer to the  $\zeta$  question is negative, we let  $\dot{\mathbf{t}}_{B_\zeta^{i+1}} \stackrel{\text{def}}{=} 1$  and choose  $q_{\zeta+1}^{i+1} = ({}^1q_{\zeta+1}^{i+1}, {}^2q_{\zeta+1}^{i+1})$  in  $M$  exemplifying the negative answer.

At any rate,  $\dot{\mathbf{t}}_{B_\zeta^{i+1}}$  is a  $R_\kappa^+ * \dot{P}_\chi$ -name for an ordinal. By Claim 2.13, in  $V^{R_\kappa^+}$  there is  $\alpha_{\zeta+1}^{i+1} \geq \alpha_\zeta^{i+1}$  and purely full  $p_{\zeta+1}^{i+1} \geq p_\zeta^{i+1}$  with  $p_{\zeta+1}^{i+1} \in P_{\alpha_{\zeta+1}^{i+1}}$  such that  $\dot{\mathbf{t}}_{B_\zeta^{i+1}}$  is a  $P_{\alpha_{\zeta+1}^{i+1}} / p_{\zeta+1}^{i+1}$ -name.

For  $\zeta$  limit, let  $\alpha_\zeta^{i+1} \stackrel{\text{def}}{=} \sup_{\xi < \zeta} \alpha_\xi^{i+1}$ ,  $p_\zeta^{i+1} \stackrel{\text{def}}{=} \cup_{\xi < \zeta} p_\xi^{i+1}$ , and  $q_\zeta^{i+1}$  not defined.

At the end, we let  $\alpha_{i+1} \stackrel{\text{def}}{=} \sup_{\zeta < \zeta^*(i+1)} \alpha_{\zeta+1}^{i+1}$  and  $p_{i+1}^*$  any purely full condition in  $P_{\alpha_{i+1}+1}$  with  $p_{i+1}^* \geq \bigcup_{\zeta < \zeta^*(i+1)} p_\zeta^{i+1}$ , and  $q_{i+1}^*$  such that

$$(\emptyset_{R_\kappa^+}, p_{i+1}^*) \Vdash "q_{i+1}^* \geq_{R^* * P'_{j(\alpha_{i+1})+1}} \{q_\zeta^{i+1} : \zeta \leq \zeta^*(i+1)\}."$$

Choice of  $p_i^*, q_i^*$  and  $\alpha_i$  for  $i < \chi$  limit. We let  $\alpha_i \stackrel{\text{def}}{=} \sup_{j < i} \alpha_j$  and choose  $p_i^* \in P_{\alpha_i+1}$  purely full so that  $p_i^* \geq \cup_{j \leq i} p_j^*$ , and if  $\text{cf}(i) \geq \theta$ , then

$$p_i^*(\alpha_i) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \tilde{B}[G_{P_{\alpha_i}}] : \tilde{B} \text{ is a } P_{\alpha_i}/(p_i^* \restriction \alpha_i)\text{-name for a subset of } \kappa \\ \text{and } \mathbf{t}_B[G_{P_{\alpha_i}}] = 1 \end{array} \right\}.$$

It follows by the construction of Laver's diamond and standard arguments about elementary embeddings and master conditions that

$$p_i^* \restriction \alpha_i \Vdash_{P_{\alpha_i}} "p_i^*(\alpha_i) \in \text{NUF}."$$

Then we can choose  $q_i^*$  so that  $(\emptyset_{R_\kappa^+}, p_i^*, q_i^*) \geq (\emptyset_{R_\kappa^+}, p_j^*, q_j^*)$  for all  $j < i$  and  $q_i^* \geq \{\mathbf{j}(r) : r \in G_{P_{\alpha_i}}\}$ , which is again possible by the observation at the beginning of the proof.  $\star_{1.17}$

**Conclusion 1.19.** In  $V_1$ , if  $\bar{Q} \in \mathcal{K}_\theta^+$  and  $\langle p_i^* : i < \chi \rangle$  is as guaranteed by Main Claim 1.17, letting  $\mathcal{D}_i \stackrel{\text{def}}{=} p_i^*(\alpha_i)$ , it follows by Observation 1.13(5) that

$$\bar{P} = \langle P_{\alpha_i}/(p_i^* \restriction \alpha_i), \mathcal{Q}_{\mathcal{D}_i}^{\alpha_i} : i < \chi \rangle$$

is an iteration with  $(< \kappa)$ -supports of  $(< \kappa)$ -directed-closed  $\theta$ -cc forcing. In addition, there is a club  $C$  of  $\chi$  with the property that in  $V_1^P$

$$\langle \mathcal{D}_i : i \in C \ \& \ \text{cf}(i) \geq \theta \rangle$$

is an increasing sequence of normal filters over  $\kappa$ , with

$$[i \in C \ \& \ \text{cf}(i) \geq \theta] \implies P_{\alpha_i}/(p_i^* \restriction \alpha_i) \Vdash "\mathcal{D}_i \text{ is an ultrafilter over } \kappa".$$

If  $\delta < \chi$  satisfies  $\text{cf}(\delta) > \kappa$  then  $\cup_{i < \delta} p_i^*$  forces over  $P_{\alpha_\delta}$  that  $\bigcup_{i < \delta} \mathcal{D}_i$  is an ultrafilter over  $\kappa$  which is generated by  $\text{cf}(\delta)$  sets.



**Definition 1.20.** (In  $V^{R_\kappa^+}$ ) Given  $\bar{Q} = \langle P_i, \mathcal{Q}_i, \mathcal{A}_i : i < \chi \rangle \in \mathcal{K}_\theta^+$ .

We say that  $\bar{Q}$  is *fitted* iff there is a continuous increasing sequence  $\langle \alpha_i : i < \chi \rangle$  of ordinals  $< \chi$ , and a sequence  $\langle p_i^* : i < \chi \rangle$  of purely full conditions with  $p_i^* \in P_{\alpha_i+1}$ , such that letting  $\mathcal{D}_i \stackrel{\text{def}}{=} p_{i(\alpha_i)}^*$ ,

$$\langle P_{\alpha_i+1}/(p_i^* \restriction \alpha_i), \mathcal{Q}_{\mathcal{D}_i}^{\alpha_i} : i < \chi \rangle$$

is an iteration with  $(< \kappa)$ -supports of  $(< \kappa)$ -directed-closed  $\theta$ -cc forcing, and

$$\text{cf}(i) \geq \theta \implies \Vdash_{P_{\alpha_i+1}/(p_i^* \restriction \alpha_i)} \text{“}\mathcal{A}_i \in \mathcal{D}_i\text{”}.$$

**Crucial Claim 1.21.** (In  $V^{R_\kappa^+}$ ) The following is a sufficient condition for  $\bar{Q} \in \mathcal{K}_\theta^+$  to be fitted:

There is a definition **R** such that:

- (1) for every forcing  $\mathbb{P}$  with  $|\mathbb{P}| \leq \Upsilon$  in  $V^\mathbb{P}$  and a  $\mathbb{P}$ -name  $\mathcal{D}$  of a normal ultrafilter on  $\kappa$  we have that  $\mathbf{R}[\mathbb{P}, \mathcal{D}]$  is a  $\mathbb{P}$ -name of a forcing notion of cardinality  $\leq \Upsilon$ ,
- (2) for every purely full  $p \in P_\chi$  and  $i \in \text{Dom}(p)$ , we have that

$$p \restriction i \Vdash \text{“}p(i) = \mathcal{Q}_{p(i)}^i = \mathbf{R}[P_i/(p \restriction i), p(i)]\text{”},$$

- (3) there is a definition  $f$  that for every forcing  $\mathbb{P}$  with  $|\mathbb{P}| \leq \Upsilon$  in  $V^\mathbb{P}$  and a  $\mathbb{P}$ -name  $\mathcal{D}$  of a normal ultrafilter on  $\kappa$  gives a  $\mathbb{P}$ -name of a function  $f_{[\mathbb{P}, \mathcal{D}]} : \mathbf{R}[\mathbb{P}, \mathcal{D}] \rightarrow \mathcal{D}$  such that for every purely full  $p \in P_\chi$  and  $i \in \text{Dom}(p)$  it is forced by  $p \restriction i$  that:

“for every inaccessible  $\kappa' < \kappa$  and every  $\mathbf{g}$  a  $(< \kappa')$ -directed family of conditions in  $\mathbf{R}[P/(p \restriction i), p(i)]$  of size  $< \kappa$ , such that

$$r \in \mathbf{g} \implies \kappa' \in f_{[P/(p \restriction i), p(i)]}^{(r)},$$

there is  $q \geq \mathbf{g}$  such that  $q \Vdash \kappa' \in \mathcal{A}_i$ .”

**Remark 1.22.** The condition in Claim 1.21 is sufficient for the present application in §2. It may be weakened if needed for some future application. Really, the condition to use instead of it is that in item (i) of Main Claim 1.17, for all  $i$  of cofinality  $< \theta$ , we are “in the good case”, i.e. the first case of item (i). However, we wish to have a criterion which can be used without the knowledge of the proof of the Main Claim 1.17, and the condition in Claim 1.21 is one such criterion.

**Proof of the Crucial Claim.** By Conclusion 1.19 it suffices to show that under the assumptions of this Claim, in the proof of Main Claim 1.17 we can choose  $\langle \alpha_i : i < \chi \rangle$ ,  $\langle p_i^* : i < \chi \rangle$  and  $\langle q_i^* : i < \chi \rangle$  so that for every  $i$  with  $\text{cf}(i) \geq \theta$ , the answer to “the 1st question” in the choice of  $q_1^{i+1}$  is negative. The proof is by induction on such  $i$ . We use the notation of Main Claim 1.17.

Given  $i$  with  $\text{cf}(i) \geq \theta$ . Hence we have

$$p_i^*(\alpha_i) = \left\{ \tilde{B}[G_i] : \begin{array}{l} \tilde{B} \text{ a } P_{\alpha_i}/(p_i^* \restriction \alpha_i)\text{-name for a} \\ \text{subset of } \kappa \text{ and } \mathfrak{t}_{\tilde{B}} = 1 \end{array} \right\} \stackrel{\text{def}}{=} \mathcal{D}_i.$$

In  $M$  we have

$$(\emptyset_{R_\kappa^+}, p_i^*, q_i^*) \Vdash \left( \begin{array}{l} \{\mathbf{j}(r)(\mathbf{j}(\alpha_i)) : \mathbf{j}(r) \in X_i\} \text{ is } (< \kappa) - \text{directed of size } < \\ \mathbf{j}(\kappa), \kappa \text{ is inaccessible and } (\forall r)[\kappa \in \mathbf{j}(f_{\tilde{B}[P/(p_i^* \restriction \alpha_i), p_i^*(\alpha_i)]}(r))] \end{array} \right).$$

(The last statement is true by the definition of  $\mathcal{D}_i$  and  $\mathfrak{t}_{\tilde{B}}$ , no matter what  $f_{[P/p^* \restriction \alpha_i, p_i^*(\alpha_i)]}(r)$  is forced to be.)

By the assumption (3) and elementarity, applying  $\mathbf{j}$  we have that the answer to the “1st question” is negative. ★<sub>1.21</sub>

**Definition 1.23.** (In  $V^{R_\kappa^+}$ ) Given  $\theta = \text{cf}(\theta) \in (\kappa, \chi)$ . We define  $\mathcal{K}_\theta^*$  in the same way as  $\mathcal{K}_\theta^+$ , but with a freedom of choice for  $Q_0$ . Namely, to obtain the definition of  $\mathcal{K}_\theta^*$  from that of  $\mathcal{K}_\theta^+$ , we

(A) In item (6) of Definition 1.8, require  $i > 0$ ,

(B) We let  $Q_0$  be any  $(< \kappa)$ -directed-closed  $\theta$ -cc forcing notion in  $\mathcal{H}(\chi)$ .

**Claim 1.24.** (In  $V^{R_\kappa^+}$ ) Main Claim 1.17, Conclusion 1.19, Definition 1.20 and Claim 1.21 hold with  $\mathcal{K}_\theta^+$  replaced by  $\mathcal{K}_\theta^*$ .

**Proof of the Claim.** As in  $V^{R_\kappa^+ * Q_0}$ ,  $\kappa$  is still indestructibly supercompact and  $\Upsilon^\theta = \Upsilon$ . ★<sub>1.24</sub>

**Discussion 1.25.** (1) In the present application, we need to make sure that cardinals are not collapsed, so we have  $\theta = \kappa^+$  and is  $Q_{\mathcal{D}}$  chosen to have a strong version of  $\kappa^+$ -cc which is preserved by iterations with  $(< \kappa)$ -supports.

(2) Clearly, Claim 1.21 remains true if we replace the word “inaccessible” by e.g “strongly inaccessible”, “weakly compact”, “measurable”.

(3) As we shall see in section 2, the point of dealing with a fitted member of  $\mathcal{K}_\theta^+$  is to be able to control the Prikry names in the forcing that will be performed after the iteration extracted from  $\mathcal{K}_\theta^+$ , namely the Prikry forcing over  $\cup_{i < \delta} \mathcal{D}_i$  for some  $\delta$ . The point of  $\mathcal{A}_i$  is to give us a control of this ultrafilter in the appropriate universe. With this in mind, we could use Claim 1.21 to represent our results in the axiomatic form, and there is also an equivalent game-theoretic representation. As it is not entirely clear that Claim 1.21 is the best sufficient condition for fittedness, we have decided not to formulate any axioms here.

## 2 Universal graphs

**Theorem 2.1.** Assume that it is consistent that a supercompact cardinal  $\kappa$  exists, and let  $\Upsilon$  and  $\chi$  be such that  $\Upsilon^+ = \chi$  and  $\Upsilon^{\kappa^+} = \Upsilon$ .

Then it is consistent to have a singular strong limit cardinal  $\mu$  of cofinality  $\omega$  with  $2^{\mu^+} = \chi > \mu^{++}$ , on which there are  $\mu^{++}$  graphs of size  $\mu^+$  which are universal for the graphs of size  $\mu^+$ .

**Proof.** We start with a universe  $V$  in which  $\mu$ ,  $\Upsilon$  and  $\chi$  satisfy Hypothesis 1.3, with  $\mu$  in place of  $\kappa$  and  $\theta = \mu^+$ . Let  $R_\kappa^+$  be the forcing described in Definition 1.5. We work in  $V^{R_\kappa^+}$ , which we start calling  $V$  from this point on. As we shall not use  $R_\kappa^+$  any more, we free the notation  $\tilde{R}_\alpha$  to be used with a different meaning in this section.

**Definition 2.2.** Let  $Q_0$  be the Cohen forcing which makes  $2^{\mu^+} = \Upsilon$  by adding  $\Upsilon$  distinct  $\mu^+$ -branches  $\{\eta_\alpha : \alpha < \Upsilon\}$  to  $(\mu^+ > 2)^V$  by conditions of size  $\leq \mu$ . Let  $V_0 \stackrel{\text{def}}{=} V[G_{Q_0}]$ .

**Notation 2.3.** If  $\kappa$  is measurable and  $\mathcal{D}$  is a normal ultrafilter on  $\kappa$ , let  $\text{Pr}(\mathcal{D})$  denote the Prikry forcing for  $\mathcal{D}$ .

**Discussion 2.4.** The idea of the proof is to embed “ $\mathcal{D}$ -named graphs” into a universal graph. We use an iteration of forcing to achieve this. As we intend to perform a Prikry forcing at the end of iteration, we need to control the names of graphs that appear after the Prikry forcing, so one worry is that there would be too many names to take care of by the bookkeeping. Luckily, we shall not be dealing with all such names, but only with those for which we are sure that they will actually be used at the end. This is achieved by building the ultrafilter that will serve for the Prikry forcing, as the union of filters that appear during the iteration. To this end, for every relevant  $\mathcal{D}$  we also force a set  $\tilde{A}$  that will in some sense be a “diagonal intersection” of  $\mathcal{D}$ , so its membership in the intended ultrafilter will guarantee that that ultrafilter contains  $\mathcal{D}$  as a subset.

**Definition 2.5.** Suppose  $V' \supseteq V_0$  is a universe in which  $2^{\kappa^+} \leq \Upsilon$ , while  $\kappa \leq \mu$  is measurable and  $\mathcal{D}$  is a normal ultrafilter over  $\kappa$ . Working in  $V'$ , we define a forcing notion  $Q = Q_{\mathcal{D}} \stackrel{\text{def}}{=} Q_{\mathcal{D}, \kappa}^{V'}$ , as follows.

Let  $\tilde{M} = \langle \tilde{M}_\alpha = \langle \kappa^+, \tilde{R}_\alpha \rangle : \alpha < \Upsilon \rangle$  list without repetitions all canonical (in the usual sense)  $\text{Pr}(\mathcal{D})$ -names for graphs on  $\kappa^+$ . For definiteness we pick the first such list in the canonical well-order of  $\mathcal{H}(\chi)$ . Elements of  $Q$  are of the form

$$p = \langle A^p, B^p, u^p, \bar{f}^p = \langle f_\alpha^p : \alpha \in u^p \rangle \rangle,$$

where

- (i)  $A^p \in [\kappa]^{<\kappa}$ ,
- (ii)  $B^p \in \mathcal{D} \cap \mathcal{P}([\kappa \setminus (\text{Sup}(A^p))])$ ,
- (iii)  $u^p \in [\Upsilon]^{<\kappa}$ ,
- (iv) For  $\alpha \in u^p$ , we have that  $f_\alpha^p$  is a partial one-to-one function from  $\kappa^+$  with  $|\text{Dom}(f_\alpha^p)| < \kappa$ , mapping  $\zeta \in \text{Dom}(f_\alpha^p)$  to an element of  $\{\eta_\alpha \restriction \zeta\} \times \kappa$ ,
- (v) For  $\alpha, \beta \in u^p$ , for every  $x', x'', y', y''$ , if

$$f_\alpha^p(x') = f_\beta^p(y') \neq f_\alpha^p(x'') = f_\beta^p(y''),$$

then for every  $w \in [A^p]^{<\aleph_0}$

$$\langle w, B^p \rangle \Vdash_{\text{Pr}(\mathcal{D})} "M_\alpha \models R_\alpha(x', x'') \text{ iff } M_\beta \models R_\beta(y', y'')",$$

and for every  $w \in [A^p]^{<\aleph_0}$  the condition  $\langle w, B^p \rangle$  decides in the Prikry forcing for  $\mathcal{D}$  if  $M_\alpha \models R_\alpha(x', x'')$

We define the order on  $Q$  by letting  $p \leq q$  (here  $q$  is a stronger condition) iff

- (a)  $A^p$  is an initial segment of  $A^q$ ,
- (b)  $A^q \setminus A^p \subseteq B^p$ ,
- (c)  $B^p \supseteq B^q$ ,
- (d)  $u^p \subseteq u^q$ ,
- (e) For  $\alpha \in u^p$ , we have  $f_\alpha^p \subseteq f_\alpha^q$ .

**Claim 2.6.** Suppose that  $Q = Q_{\mathcal{D}, \kappa}^{V'}$  is defined as in Definition 2.5. Then in  $V'$ :

- (1)  $Q$  is a separative partial order.
- (2) Suppose that  $G$  is  $Q$ -generic over  $V'$ , and let in  $V'[G]$

$$A^* \stackrel{\text{def}}{=} \bigcup \{A : (\exists B, u, \bar{f})[\langle A, B, u, \bar{f} \rangle \in G]\}.$$

Then  $A^* \in [\kappa]^\kappa$  and  $A^* \subseteq^* B$  for every  $B \in \mathcal{D}$ .

(3) For  $\alpha < \Upsilon$  and  $a \in \kappa^+$ , the set

$$\mathcal{K}_{a,\alpha} \stackrel{\text{def}}{=} \{p \in Q : \alpha \in u^p \text{ \& } a \in \text{Dom}(f_\alpha^p)\}$$

is dense open in  $Q$ .

**Proof of the Claim.** (1) Routine checking.

(2) For  $\alpha < \kappa$ , the set

$$\mathcal{I}_\alpha \stackrel{\text{def}}{=} \{p \in Q : (\exists \beta \geq \alpha)[\beta \in A^p]\}$$

is dense open, hence  $A^* \in [\kappa]^\kappa$ . For  $B \in \mathcal{D}$  the set

$$\mathcal{J}_B \stackrel{\text{def}}{=} \{p \in Q : B^p \subseteq B\}$$

is dense open. If  $p \in \mathcal{J}_B \cap G$ , then for any  $q \in G$  with  $q \geq p$  we have  $A^q \setminus A^p \subseteq B^p$ . Hence  $A^* \setminus B \subseteq A^p$ .

(3) Given  $p \in Q$ , clearly there is  $q \geq p$  with  $\alpha \in u^q$ . Without loss of generality  $\alpha \in u^p$  and  $a \notin \text{Dom}(f_\alpha^p)$ . Applying the Prikry Lemma, for every  $b \in \text{Dom}(f_\alpha^p)$  and  $w \in [A^p]^{<\aleph_0}$ , there is  $B_{w,b} \subseteq B^p$  with  $B_{w,b} \in \mathcal{D}$  and such that

$$(w, B_{w,b}) \Vdash_{\text{Pr}(\mathcal{D})} \text{“}\dot{M}_\alpha \models b \dot{R}_\alpha a\text{”}.$$

Choose  $\gamma < \kappa$  such that  $(\eta_\alpha \restriction a, \gamma) \notin \bigcup_{\beta \in u^p} \text{Rang}(f_\beta^p)$ , which is possible as for every relevant  $\beta$  we have  $|\text{Dom}(f_\beta^p)| < \kappa$ . Now we define  $q$  by letting  $A^q \stackrel{\text{def}}{=} A^p$ ,  $B^q \stackrel{\text{def}}{=} \bigcap \{B_{w,b} : w \in [A^p]^{<\aleph_0} \text{ \& } b \in \text{Dom}(f_\alpha^p)\}$ ,  $u^q \stackrel{\text{def}}{=} u^p$  and

$$f_\beta^q \stackrel{\text{def}}{=} \begin{cases} f_\beta^p & \text{if } \beta \neq \alpha \\ f_\alpha^p \cup \{(a, (\eta_\alpha \restriction a, \gamma))\} & \text{otherwise.} \end{cases}$$

★<sub>2.6</sub>

**Notation 2.7.** Suppose that  $Q$  is as in Claim 2.6. For  $\alpha < \Upsilon$  let

$$\dot{f}_\alpha \stackrel{\text{def}}{=} \cup \{f_\alpha^p : \alpha \in u^p \text{ \& } p \in G_Q\}.$$

**Definition 2.8.** (Shelah, [Sh 80]) Let  $\lambda \geq \aleph_0$  be a cardinal. A forcing notion  $P$  is said to be *stationary  $\lambda^+$ -cc* iff for every  $\langle p_\alpha : \alpha < \lambda^+ \rangle$  in  $P$ , there is a club  $C \subseteq \lambda^+$  and a regressive  $f : \lambda^+ \rightarrow \lambda^+$  such that for all  $\alpha, \beta \in C$ ,

$$[\text{cf}(\alpha) = \text{cf}(\beta) = \lambda \ \& \ f(\alpha) = f(\beta)] \implies p_\alpha, p_\beta \text{ are compatible.}$$

**Theorem 2.9.** Shelah ([Sh 80]) Suppose that  $\lambda^{<\lambda} = \lambda \geq \aleph_0$ . Iterations with  $(< \lambda)$ -support of  $(< \lambda)$ -directed-closed stationary  $\lambda^+$ -cc forcing, are  $(< \lambda)$ -directed-closed and satisfy stationary  $\lambda^+$ -cc.

**Claim 2.10.** Suppose that  $Q$  is as in Claim 2.6. Then  $Q$  is  $(< \kappa)$ -directed-closed and satisfies stationary  $\kappa^+$ -cc.

**Proof of the Claim.** First suppose that  $i^* < \kappa$  and  $\{p_i : i < i^*\}$  is directed. For  $i < i^*$  let  $p_i \stackrel{\text{def}}{=} \langle A^i, B^i, u^i, \bar{f}^i \rangle$ . We define  $A \stackrel{\text{def}}{=} \bigcup_{i < i^*} A^i$ ,  $B \stackrel{\text{def}}{=} \bigcap_{i < i^*} B^i$ ,  $u \stackrel{\text{def}}{=} \bigcup_{i < i^*} u^i$ , and for  $\alpha \in u$  we let  $f_\alpha \stackrel{\text{def}}{=} \bigcup_{i < i^*} f_\alpha^i$ . It is easily verified that this defines a common upper bound of all  $p_i$ .

Hence  $Q$  is  $(< \kappa)$ -directed-closed. Now we shall prove that it is  $\kappa^+$ -stationary-cc. Let  $\langle p_i : i < \kappa^+ \rangle$  be given, where each  $p_i = \langle A^i, B^i, u^i, \bar{f}^i \rangle$ .

There is a stationary  $S \subseteq S_{\kappa^+}^{\kappa^+}$  and  $A^\# \in [\kappa]^{<\kappa}$  and  $\sigma, \tau < \kappa$  such that for all  $i \in S$  we have  $A^i = A^\#$  and  $|u^i| = \sigma$ , and  $|\bigcup \{\text{Dom}(f_\alpha^i) : \alpha \in u^i\}| = \tau$ . For  $i \in S$ , let  $\zeta_i \stackrel{\text{def}}{=} \sup \bigcup_{\alpha \in u^i} \text{Dom}(f_\alpha^i)$ , hence  $\text{cf}(\zeta_i) < \kappa$ . So

$$E \stackrel{\text{def}}{=} \{j < \kappa^+ : \text{cf}(j) \geq \kappa \implies (\forall i < j)[\zeta_i < j]\}$$

is a club of  $\kappa^+$ . Let  $S_1 \stackrel{\text{def}}{=} S \cap E$ .

Let  $\theta \stackrel{\text{def}}{=} \sigma + \tau + |A^\#|$ , so  $\theta < \kappa$ . For  $i \in S_1$  let  $u^i \stackrel{\text{def}}{=} \{\alpha_s^i : s < \sigma\}$  be an increasing enumeration. For every such  $i$ , we define a model  $M_i$  with universe  $\kappa^+$ , relations  $\mathbf{R}_{w,s}^i$  for  $w \in [A^\#]^{<\aleph_0}$  and  $s < \sigma$ , and (partial) functions  $g_s^i$  from  $\kappa^+$  to  $\kappa$ , for  $s < \sigma$ . This model is defined by letting

$$(\zeta, \xi) \in \mathbf{R}_{w,s}^i \text{ iff } [\zeta, \xi \in \text{Dom}(f_{\alpha_s^i}^i) \ \& \ (w, B^i) \Vdash_{\text{Pr}(\mathcal{D})} \text{“}\zeta R_{\alpha_s^i} \xi\text{”}],$$

and

$$g_s^i(\zeta) = \gamma \text{ iff } f_{\alpha_s^i}^i(\zeta) = (\eta_{\alpha_s^i}, \gamma).$$

Note that always  $|\mathbf{R}_{w,s}^i| \leq \theta$  and  $|g_s^i| \leq \theta$ .

Now let  $X$  be the set of all isomorphism types of models with their universe an ordinal  $< \kappa^+$  and  $\leq \theta$  relations and functions, each of cardinality  $\leq \theta$ . Hence  $|X| = \kappa^+$ , let  $X \stackrel{\text{def}}{=} \{t_i : i < \kappa^+\}$ . Now note that there is a club  $C$  of  $\kappa^+$  such that for every  $j \in C \cap S_\kappa^+$ , types of all models with universe  $< j$  and  $\leq \theta$  relations of functions, each of cardinality  $\leq \theta$ , are enumerated in  $X$  with an index  $< j$ . Let  $S_2 \stackrel{\text{def}}{=} C \cap S_1$ .

For  $i \in S_2$ , let  $h(i) = l$  iff  $t_l$  is the type of  $M_i \upharpoonright \sup(i \cap \bigcup_{\alpha \in u^i} \text{Dom}(f_\alpha^i)) + 1$ . Hence  $h$  is regressive on  $S_2$ , so there is a stationary subset  $S_3$  of  $S_2$  such that  $h$  is constant on  $S_3$ .

It is easily verified that  $p_i, p_j$  are compatible for every  $i, j \in S_3$ . ★<sub>2.10</sub>

**Observation 2.11.** Suppose that  $\mathcal{D}$  is a normal ultrafilter over  $\kappa$  and  $Q$  is a forcing notion such that

$$\Vdash_Q \text{ “}\mathcal{D} \subseteq \mathcal{D}' \text{ and } \mathcal{D}' \text{ is a normal ultrafilter over } \kappa\text{”}.$$

Then  $\text{Pr}(\mathcal{D}) <_{fQ} \text{Pr}(\mathcal{D}')$ , where  $f$  is the embedding given by

$$f((a, A)) \stackrel{\text{def}}{=} (\emptyset_Q, (a, A)).$$

**Definition 2.12.** Suppose that  $Q$  is as in Claim 2.6, while  $Q <_{\circ} P$ , and  $\mathcal{D}'$  is a  $P$ -name of a normal ultrafilter over  $\kappa$ , extending  $\mathcal{D} \cup \{A^*\}$ . For  $\alpha < \Upsilon$  we define  $\mathcal{G}r_\alpha^{\mathcal{D}'}$ , intended to be a name for a graph on  $\{\eta_\alpha \upharpoonright \zeta : \zeta < \kappa^+\} \times \kappa$  (see Claim 2.13 below), defined by letting for  $y', y'' \in \{\eta_\alpha \upharpoonright \zeta : \zeta < \kappa^+\} \times \kappa$ ,

$$\begin{aligned} y' \mathcal{R} y'' & \text{ iff for some } \langle p, \langle w, B^p \rangle \rangle \in \mathcal{G} \text{ with } \alpha \in u^p, p \in Q \text{ and } [w] \in [A^p]^{<\aleph_0} \\ & \text{ and some } x', x'' \in \text{Dom}(f_\alpha^p) \\ & \text{ we have } f_\alpha^p(x') = y' \text{ and } f_\alpha^p(x'') = y'', \\ \text{AND } \langle w, B^p \rangle & \Vdash_{\text{Pr}(\mathcal{D})} \text{ “}\mathcal{M}_\alpha \models \mathcal{R}_\alpha(x', x'')\text{”}. \end{aligned}$$

**Claim 2.13.** Suppose  $Q$  is as in Claim 2.6, while  $Q <_{\circ} P$ , and  $\mathcal{D}'$  is a  $P$ -name of a normal ultrafilter over  $\kappa$ , extending  $\mathcal{D} \cup \{A^*\}$  (equivalently,  $A^* \in \mathcal{D}'$ ).

Then

$$\langle \emptyset, \langle \emptyset, A^* \rangle \rangle \Vdash_{P * \text{Pr}(\mathcal{D}')} \text{ “}f_\alpha \text{ is an embedding of } \mathcal{M}_\alpha \text{ into } \mathcal{G}r_\alpha^{\mathcal{D}'}\text{”}.$$



**Proof of the Claim.** Let  $G$  be  $P * \text{Pr}(\mathcal{D}')$ -generic with  $\langle \emptyset, \langle \emptyset, \underline{A}^* \rangle \rangle \in G$  and suppose that  $x', x''$  are such that  $M_\alpha \models R_\alpha(x', x'')$  in  $V[G]$ . Let  $\langle p^+, \langle w, \underline{A}' \rangle \rangle$  be a condition in  $G$  that forces this. Without loss of generality, we have

$$\langle p^+, \langle w, \underline{A}' \rangle \rangle \geq \langle \emptyset, \langle \emptyset, \underline{A}^* \rangle \rangle.$$

In particular,  $p^+ \Vdash_P "w \in [\underline{A}^*]^{<\aleph_0}"$ . Considering  $P$  as  $Q * P/Q$ , let us write  $\langle p^+, \langle w, \underline{A}' \rangle \rangle$  as  $\langle p, p', \langle w, \underline{A}' \rangle \rangle$ . As  $\underline{A}^*$  is a  $Q$ -name, by extending  $p^+$  if necessary, we may assume that  $A^p \supseteq w$ , and then using the density of  $\mathcal{K}_{x', \alpha}$  and  $\mathcal{K}_{x'', \alpha}$ , we may also assume that  $\alpha \in u^p$  and  $x', x'' \in \text{Dom}(f_\alpha^p)$ . By extending further, we may assume that  $p^+ \Vdash "A' \subseteq B^p"$ . Then  $\langle p^+, \langle w, \underline{A}' \rangle \rangle \geq \langle p, \langle w, B^p \rangle \rangle$ , hence the latter is in  $G$ . Since  $p \Vdash_P "\langle w, b^p \rangle \Vdash_{\text{Pr}(\mathcal{D})} R_\alpha(x', x')"$ , it must be that  $\langle w, B^p \rangle \Vdash_{\text{Pr}(\mathcal{D})} "M_\alpha \models R_\alpha(x', x')"$ . Hence in  $V[G]$  we have that

$$y' = f_\alpha(x')Ry'' = f_\alpha(x'').$$

On the other hand, suppose that in  $V[G]$  we have  $y' = f_\alpha(x')Ry'' = f_\alpha(x'')$  and let  $\langle p, \langle w, B^p \rangle \rangle$  exemplify this. In particular,  $\langle w, B^p \rangle$  forces in  $\text{Pr}(\mathcal{D})$  that " $M_\alpha \models R_\alpha(x', x'')$ ", and since  $\langle p, \langle w, B^p \rangle \rangle \in G$ , we have that  $R_\alpha(x', x'')$  holds in  $V[G]$ .

As it is easily seen that each  $f_\alpha$  is forced to be 1-1 and total, this finishes the proof. ★<sub>2.13</sub>

**Claim 2.14.** Suppose that  $Q$  and  $\mathcal{D}'$  are as in Claim 2.13, while  $G$  is  $Q$ -generic over  $V'$  and  $2^\kappa = \kappa^+$  holds in  $V'$ . Further suppose that  $H$  is a  $\text{Pr}(\mathcal{D}')$ -generic filter over  $V'[G]$  with  $\langle \emptyset, A^* \rangle \in H$ .

Then in  $V'[G][H]$ , there is a graph  $Gr^*$  of size  $\kappa^+$  such that for every  $\text{Pr}(\mathcal{D})$ -generic filter  $J$  over  $V'$ , every graph of size  $\kappa^+$  in  $V'[J]$  is embedded into  $Gr^*$ .

**Proof of the Claim.** Define  $Gr^*$  on  $\cup_{\alpha < \Upsilon} \{\eta_\alpha \restriction \zeta : \zeta < \kappa^+\} \times \kappa$ , hence  $|Gr^*| = \kappa^+$ , by our assumptions on  $V'$ . We let

$$Gr^* \models "(\eta_\alpha \restriction \zeta, i)R(\eta_\alpha \restriction \xi, j)" \text{ iff } Gr_\alpha^{\mathcal{D}'} \models "(\eta_\alpha \restriction \zeta, i)R(\eta_\alpha \restriction \xi, j)".$$

Then  $Gr^*$  is a well defined graph, as follows by the definition of  $Q$ .

Given  $M$  a graph on  $\kappa^+$  in  $V'[J]$ , there is  $\alpha$  such that  $M = \tilde{M}_\alpha[G][J]$ , hence  $M$  embeds into  $Gr_\alpha^{\mathcal{D}'}$ , which is a subgraph of  $Gr^*$ .  $\star_{2.14}$

We thank Charles Morgan for permitting us to use the following argument he showed us:

**Claim 2.15.** Let  $\mathcal{D}$  be a normal ultrafilter over  $\kappa$  and  $A \in \mathcal{D}$ . Suppose that  $G$  is  $\text{Pr}(\mathcal{D})$ -generic filter over  $V$ . Then there is some  $G'$  which is  $\text{Pr}(\mathcal{D})$ -generic over  $V$  and such that  $(\emptyset, A) \in G'$  while  $V[G] = V[G']$ .

**Proof of the Claim.** Let  $x = x_G = \cup\{s : (\exists B \in \mathcal{D})(s, B) \in G\}$ , so

$$G = G_x = \{(s, B) \in \text{Pr}(\mathcal{D}) : s \subseteq x_G \subseteq s \cup B\}.$$

Now we use the Mathias characterisation of Prikry forcing, which says that for an infinite subset  $x$  of  $\kappa$  we have that  $G_x$  is  $\text{Pr}(\mathcal{D})$ -generic over  $V$  iff  $x_G \setminus B$  is finite for all  $B \in \mathcal{D}$ . Hence  $x \setminus A$  is finite. Let  $y = x_G \cap A$ , so an infinite subset of  $\kappa$  which clearly satisfies that  $y \setminus B$  is finite for all  $B \in \mathcal{D}$ . Let  $G' = G_y$ , so  $G'$  is  $\text{Pr}(\mathcal{D})$ -generic over  $V$  and  $(\emptyset, A) \in G'$ . We have  $V[G'] \subseteq V[G]$  because  $y \in G$  and  $V[G] \subseteq V[G']$  because  $x \setminus y$  is finite.

$\star_{2.15}$

**Conclusion 2.16.** Suppose that  $Q$ ,  $\mathcal{D}'$ ,  $G$  and  $V'$  are as in Claim 2.14 and  $H$  is a  $\text{Pr}(\mathcal{D}')$ -generic filter over  $V'[G]$ . Then the conclusion of Claim 2.14 holds in  $V'[G][H]$ .

**Proof.** Since  $\mathcal{E} \stackrel{\text{def}}{=} \{\langle s, B \rangle : B \subseteq A^*\}$  is in  $V'[G]$  and dense in  $\text{Pr}(\mathcal{D}')$ , there is  $\langle s, B \rangle \in H \cap \mathcal{E}$ . Let  $H^* \stackrel{\text{def}}{=} \{\langle t, C \rangle : t \supseteq s\}$ . Then  $\langle s, A^* \rangle \in H^*$  and  $H^*$  is  $\text{Pr}(\mathcal{D}')/s$ -generic over  $V'[G]$  with  $V'[G][H^*] = V'[G][H]$ . As  $\langle s, A^* \rangle$  forces in  $\text{Pr}(\mathcal{D}')/s$  exactly the same statements as  $\langle \emptyset, A^* \rangle$  does in  $\text{Pr}(\mathcal{D}')$ , the conclusion follows by Claim 2.14.  $\star_{2.16}$

**Claim 2.17.** Suppose that  $\bar{Q} = \langle P_i, Q_i, A_i : i < \chi \rangle \in \mathcal{K}_{\kappa^+}^*$  is given by determining  $Q_0$  as in Definition 2.2 and defining  $Q_{\mathcal{D}}^i = Q_{\mathcal{D}, \kappa}^{V[G_{P_i}]}$  as defined in Definition 2.5, with  $\kappa$  replaced by  $\mu$ , and  $A_i = \tilde{A}_i^*$  where  $\tilde{A}_i^*$  was defined in Claim 2.6(2).

Then  $\bar{Q}$  is fitted.

**Proof of the Claim.** We shall take  $\tilde{R}$  to be defined by Definition 2.5. By Claim 1.21, it suffices to give a definition of  $f$  satisfying the requirements of that Claim. Suppose that  $P, \mathcal{D}$  is such that  $\tilde{R}[P, \mathcal{D}]$ , working in  $V^P$  we define

$$f = f_{[P, \mathcal{D}]} : Q_{\mathcal{D}} = R_{[P, \mathcal{D}]} \rightarrow \mathcal{D}$$

by letting  $f(p) \stackrel{\text{def}}{=} B^p$  for  $p = (A^p, B^p, u^p, \bar{f}^p)$ . We check that this definition is as required. So suppose that  $\kappa' < \kappa$  is inaccessible and  $\mathbf{g}$  is a  $(< \kappa')$ -directed family of conditions in  $Q_{\mathcal{D}}$  with the property that for all  $p \in \mathbf{g}$  we have  $\kappa' \in B^p$ . We define  $r$  by letting

$$A^r \stackrel{\text{def}}{=} \bigcup_{p \in \mathbf{g}} A^p \cup \{\kappa'\}, B^r \stackrel{\text{def}}{=} \bigcap_{p \in \mathbf{g}} B^p \setminus \{\kappa'\}, u^r \stackrel{\text{def}}{=} \bigcup_{p \in \mathbf{g}} u^p,$$

and for  $\alpha \in u^r$ , we let  $f_{\alpha}^r \stackrel{\text{def}}{=} \bigcup_{p \in \mathbf{g} \ \& \ \alpha \in u^p} f_{\alpha}^p$ . It is easy to check that this condition is as required.  $\star_{2.17}$

**Remark 2.18.** The inaccessibility of  $\kappa'$  was not used in the Proof of Claim 2.17.

*Proof of the Theorem finished.*

To finish the proof of the Theorem, in  $V_0$  let  $\bar{Q}$  be as in Claim 2.17. By Claim 2.17 and the definition of fittedness, we can find sequences  $\langle p_i^* : i < \chi \rangle$  and  $\langle \alpha_i : i < \chi \rangle$  witnessing that  $\bar{Q}$  is fitted. Let  $\mathcal{D}_i \stackrel{\text{def}}{=} p_i^*(\alpha_i)$  for  $i < \chi$ . If we force in  $V_0$  by

$$P^* \stackrel{\text{def}}{=} \lim \langle P_{\alpha_i} / (p_i^* \restriction \alpha_i), \mathcal{Q}_{\mathcal{D}_i} : i < \chi \rangle,$$

we obtain a universe  $V^*$  in which  $\langle \mathcal{D}_i : \text{cf}(i) = \mu^+ \rangle$  is an increasing sequence of normal filters over  $\mu$ , and  $\mathcal{D} \stackrel{\text{def}}{=} \bigcup_{i \in S_{\mu^+}^{\chi}} \mathcal{D}_i$  is a normal ultrafilter over  $\mu$ . For, in  $V^{P_{\alpha_i} / (p_i^* \restriction \alpha_i)}$ , we have that  $\mathcal{D}_i$  is an ultrafilter over  $\mu$ , and  $\text{cf}(\chi) > \mu$ , while the iteration is with  $(< \mu)$ -supports and  $\mu^{<\mu} = \mu$ . Hence every subset if  $\mu$  in  $V^*$  appears as an element of  $V^{P_{\alpha_i} / (p_i^* \restriction \alpha_i)}$  for some  $i$ , and so  $\mathcal{D}$  is an ultrafilter.

Also, for every  $i \in S_{\mu^+}^\chi$  we have that  $A_i^* \in \mathcal{D}$ . Let  $\mathcal{D}$  be a  $P^*$ -name for  $\mathcal{D}$  of  $V^*$ . Let

$$E \stackrel{\text{def}}{=} \left\{ \delta < \chi : \begin{array}{l} (\forall \alpha < \delta)(\exists \beta \in (\alpha, \delta))[\alpha_\beta = \beta] \text{ and} \\ \mathcal{D} \cap \mathcal{P}(\kappa)^{V_0^{P_\beta}} \text{ is a } P_\beta/(p_\beta \restriction \beta)\text{-name} \\ \text{and } p_{\beta+1}(\beta) = \mathcal{D} \cap \mathcal{P}(\kappa)^{V_0^{P_\beta}} \end{array} \right\}.$$

Hence  $E$  is a club of  $\chi$ . Let  $\delta \in E \cap S_{\mu^{++}}^\chi$  be larger than  $\mu^{+++}$ . Force with  $P^* \restriction \delta$ , so obtaining  $V_1$  in which  $2^{\mu^+} \geq 2^\kappa \geq \mu^{+++}$ , as each coordinate of  $P^* \restriction \delta$  adds a subset of  $\mu$ , and cardinals are preserved. In  $V_1$  force with the Prikry forcing for  $\mathcal{D}_\delta \stackrel{\text{def}}{=} \bigcup_{i \in S_{\mu^+}^\delta} \mathcal{D}_i$ . Let  $W \stackrel{\text{def}}{=} V_1[\text{Pr}(\mathcal{D}_\delta)]$ . For  $i \in S_{\mu^+}^\delta$ , let  $\text{Gr}_i^*$  be a graph obtained in  $W$  satisfying the conditions of Conclusion 2.16 with  $\mathcal{D}_\delta$  in place of  $\mathcal{D}'$  and  $\mathcal{D}_i$  in place of  $\mathcal{D}$ . Let  $C$  be a club of  $\delta$  of order type  $\mu^{++}$ , and let  $g$  be its increasing enumeration.

We claim that  $W$  is as required, and that

$$\{\text{Gr}_{g(i)}^* : i < \mu^{++} \text{ \& } \text{cf}(g(i)) = \mu^+\}$$

are universal for graphs of size  $\mu^+$ . Clearly the cofinality of  $\mu$  in  $W$  is  $\aleph_0$  and  $\mu$  is a strong limit. Suppose that  $Gr$  is a graph on  $\mu^+$  in  $W$  and let  $\mathbb{Q}Gr$  be a  $\text{Pr}(\mathcal{D}_\delta)$ -name for it. Hence, there is a  $i < \mu^{++}$  with  $\text{cf}(g(i)) = \mu^+$  such that  $\mathbb{Q}Gr$  is a  $\text{Pr}(\mathcal{D}_{g(i)})$ -name for a graph on  $\mu^+$ . The conclusion follows by the choice of  $Gr_i^*$ .  $\star_{2.1}$

**Remark 2.19.** The forcing used in [GiSh 597] also satisfies the conditions of Claim 1.21, again with  $f(p) = B^p$ .

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